MATH 320 FINAL SUMMARY

Chapter 1

Topics to know: subspaces, span, linear independence, basis, dimension.

(1) $W \subset V$ is a subspace if it has the zero vector, it is closed under vector addition and scalar multiplication;
(2) If $S \subset V$ is a set of vectors, then $Span(S)$ is the set of all linear combinations of S and is the smallest subspace of V containing S.
(3) $\{x_1, ..., x_n\}$ are linearly independent iff whenever $a_1x_1 + ...a_nx_n = \vec{0}$, then $a_1 = a_2 = ... = a_n = 0$.
(4) α is a basis for V iff it is linearly independent and $Span(\alpha) = V$.
(5) $\dim(V) = k$ if V has a basis α of size k, and in that case:
    (a) if $L \subset V$ is linearly independent, then $|L| \leq k$, and L can be extended to a basis,
    (b) if $G \subset V$, $Span(G) = V$, then $|G| \geq k$ and V has a basis $\beta \subset G$,
    (c) if $\gamma \subset V$ has size k, then γ is linearly independent iff γ spans V iff γ is a basis for V.

Suppose W_1, W_2 are two subspaces of V. Then (know how to prove that) $W_1 \cap W_2$ and $W_1 + W_2$ are subspaces, but $W_1 \cup W_2$ is usually not a subspace.
Direct products: $V = W_1 \oplus W_2$ iff $V = W_1 + W_2$ and $W_1 \cap W_2 = \{\vec{0}\}$

Chapter 2

Let $T : V \to W$ be a linear transformation, V, W vector spaces over F.

(1) The dimension theorem: $\dim(V) = nullity(T) + rank(T)$.
(2) If V, W have the same finite dimension, then T is onto iff T is one-to-one iff $\ker(T) = \{\vec{0}\}$.
(3) T is invertible iff T is one–to-one and onto iff V, W are isomorphic. Then we write $V \cong W$.
(4) If $\dim(V) = n$, then $V \cong F^n$.
(5) If $\dim(V) = \dim(W)$, then $V \cong W$.
(6) If $\dim(V) = n$ and β is a basis for V, then $[x]_\beta \in F^n$ is the **coordinate vector of x relative to** β, and $\phi_\beta : V \to F^n$ defined by $\phi_\beta(x) = [x]_\beta$ is an isomorphism (i.e. it's one-to-one, onto, and linear).
(7) If $T : V \to W$ linear, V, W finite dimensional, β is a basis for V, γ is a basis for W, then $[T]_\beta^\gamma$ is the corresponding matrix representation and for any $x \in V$,

$$[T]_\beta^\gamma [x]_\beta = [T(x)]_\gamma.$$

(8) If $T : V \to W$ and $T : W \to Z$ are linear transformation, then so is $UT : V \to Z$ defined by $UT(x) = U(T(x))$, and if α, β, γ are bases for V, U, Z, respectively, then
$$[U^\gamma_\beta][T]^\beta_\alpha = [UT]^\gamma_\alpha.$$
(9) If A is n by n and e is the standard basis for F^n, then $[L_A]_e = A$. (L_A denotes multiplication by A.)

Inverses: If $T : V \to W$ is a one-to-one, onto linear transformation, then its **inverse** is $T^{-1} : W \to V$ defined by $T^{-1}(y) = x$ iff $T(x) = y$. An $n \times n$ matrix A is **invertible** if there is an $n \times n$ matrix B, such that $AB = BA = I_n$. Then we write $B = A^{-1}$.

Lemma 1. *Let* $\dim(V) = \dim(W) = n$, *and* $T : V \to W$ *be a linear transformation,* β *any basis for* V, γ *any basis for* W. *Then* T *is invertible iff* $[T]^\gamma_\beta$ *is invertible.*

Also, in that case, if $A = [T]^\gamma_\beta$, *then* $A^{-1} = [T^{-1}]^\beta_\gamma$.

By the above, we get A is invertible iff L_A is invertible.

Recall that $I_V : V \to V$ is the identity linear transformation, i.e. $I_V(x) = x$ for all $x \in V$.

Theorem 2. *Suppose* V *is finite dimensional vector space and* β, α *are two bases for* V. *The* **change of coordinate matrix from α to β**, $Q = [I_V]^\beta_\alpha$ *is such that:*

(1) *for every vector* $x \in V$, $Q[x]_\alpha = [x]_\beta$,
(2) Q *is invertible and* $Q^{-1} = [I_V]^\alpha_\beta$,
(3) *If* $T : V \to V$ *is a linear transformation, then* $[T]_\alpha = Q^{-1}[T]_\beta Q$

Definition 3. *Suppose that* $A, B \in M_{n,n}(F)$. *We say that* A *and* B *are* **similar** *if there is an invertible matrix* Q *such that* $A = Q^{-1}BQ$. *We write* $A \sim B$

We have that (know how to prove these):

(1) $\sim$ is an equivalence relation;
(2) If $T : V \to V$ is linear, and α, β are two bases for V, then $[T]_\alpha \sim [T]_\beta$.
(3) If $A \sim B$, then $\det(A - tI_n) = \det(B - tI_n)$.

Chapter 3,4

Let $A \in M_{n,k}(F)$. Then multiplication by A is $L_A : F^k \to F^n$, and for $b \in F^n$, we can consider $Ax = b$, which is a system of n- linear equations with k unknowns. Then:

(1) the solution set to the homogeneous system $Ax = \vec{0}$ is the kernel of A (in particular, it is a subspace);

(2) the solution set to $Ax = b$ is of the form $S = \{s + k \mid k \in \ker(A), s \text{ is some fixed solution to } Ax = b\}$;
(3) $rank(A) = \dim(range(A)) = \dim($ column space of $A)$. Know how to prove the last equality. The first is the definition of rank.
(4) $\dim($ column space of $A) = \dim($ row space of $A)$, and so $rank(A) = rank(A^t)$.

Above A^t denotes the transpose of A. Note that (you can use this without proof), $(AB)^t = B^t A^t$. Also, A is **symmetric** is $A = A^t$. A is **skew-symmetric** if $A = -A^t$.

Let $A \in M_{n\times n}(F)$. We define **the determinant** by induction on n. If $n = 1$, $\det(A) = A = a_{11}$. If $n > 1$, then we can compute the determinant by expanding along any row or column:

- (expanding along the first row:) $\det(A) = \Sigma_{k=1}^n (-1)^{k+1} a_{1k} \cdot \det(A_{1k})$,
- (expanding along the ith row:) $\det(A) = \Sigma_{k=1}^n (-1)^{i+k} a_{ik} \cdot \det(A_{ik})$,
- (expanding along the jth column:) $\det(A) = \Sigma_{k=1}^n (-1)^{k+j} a_{kj} \cdot \det(A_{kj})$.

Here A_{ij} is the submatrix of A obtained from removing the i-th row and the jth column of A.

Some useful properties about determinants:

(1) the determinant is linear function of each row, when the other ones are fixed,
(2) $\det(AB) = \det(A) \cdot \det(B)$, and so $\det(AB) = \det(BA)$,
(3) $\det(A^t) = \det(A)$.

Let A be an $n \times n$ matrix. The following are equivalent:

(1) $Ax = \vec{0}$ only has the trivial solution,
(2) $Ax = \vec{b}$ always has a unique solution for any $b \in F^n$.
(3) $rank(A) = n$
(4) $\ker(A) = \{\vec{0}\}$
(5) A is invertible,
(6) A is the product of elementary matrices,
(7) A is row-reducible to I_n,
(8) $\det(A) \neq 0$.

Know what is an elementary matrix and what their determinants are.

Chapter 5 Diagonalization

If $T(x) = \lambda x$, for some non zero vector x, then x is an **eigenvector**, and λ is the corresponding **eigenvalue**. Similarly if $Ax = \lambda x$ for a non zero x.

$T : V \to V$ is **diagonalizable** iff V has a basis β of eigenvectors for T iff $[T]_\beta$ is a diagonal matrix. Similarly, a matrix $A \in M_{n,n}(F)$ is **diagonalizable** iff $A \sim D$ for a diagonal matrix D iff F^n has a basis of eigenvectors for A.

How to find if $A \in M_{n,n}(F)$ is diagonalizable:

Step 1. set the characteristic polynomial of A equal to zero: $f(t) = \det(A - tI_n) = 0$ and solve for t. The solutions are the eigenvalues.

Step 2. for each eigenvalue λ, solve $Ax = \lambda x$ for x, to find the linearly independent possible eigenvectors. That will be a basis for the **eigenspace** E_λ. Note that since λ is an eigenvalue, $\dim(E_\lambda) \geq 1$.

Step 3. Add up all of the eigenvectors for the different eigenvalues and see if you have $n$linearly independent many of them. If yes, then they are a basis for F^n and A is diagonalizable.

To find if $T : V \to V$ is diagonalizable, let e be the standard basis for V, and $A = [T]_e$. Do the above process for A.

Some nice properties of diagonal matrices: if D is diagonal with diagonal entries $d_{11}, d_{22}, ..., d_{nn}$, then:

- $\det(D)$ is the product of the d_{ii}'s.
- D^k is also a diagonal matrix with diagonal entries $d_{11}^k, d_{22}^k, ..., d_{nn}^k$.

Let $T : V \to V$ and $x \in V$. Then $W := Span(\{x, T(x), T^2(x), ...\})$ is the **T-cyclic subspace of V generated by x.** We have:

- W is the smallest T-invariant subspace containing x.
- If $\dim(W) = k$, then $\{x, T(x), ..., T^{k-1}(x)\}$ is a basis for W.

Cayley-Hamilton theorem: If$f(t)$ is the characteristic polynomial of $T : V \to V$, then $f(T) = T_0$ i.e. the zero linear transformation. Similarly, if $f(t)$ is the characteristic polynomial of a matrix $A \in M_{n,n}$, then $f(A)$ is the zero matrix.